Approximate Reasoning with Fuzzy Booleans

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Abstract

This paper introduces, in analogy to the concept of fuzzy numbers, the concept of fuzzy booleans, and examines approximate reasoning with the compositional rule of inference using fuzzy booleans. It is shown that each set of fuzzy rules is equivalent to a set of fuzzy rules with singleton crisp antecedents; in case of fuzzy booleans this set contains only two rules. It is shown that Zadeh's extension principle is equivalent to the compositional rule of inference using a complete set of fuzzy rules with singleton crisp antecedents. The results are applied to describe the use of approximate reasoning with fuzzy booleans to object-oriented design methods.

Keywords: fuzzy booleans, approximate reasoning, object-oriented design methods.

1. Introduction

In applications of approximate reasoning, one usually encodes fuzzy information into a set of fuzzy inference rules, and performs fuzzy inferences with either fuzzy or crisp input. In this paper we consider the case where the rules are crisp, and the input consists of uncertain decisions. For instance, in [1,6,7] Aksit and Marcelloni have considered application of approximate reasoning to design methods for object-oriented systems with uncertain decisions. A typical crisp rule in such a design method is:

IF an entity is relevant AND this entity is autonomous THEN this entity is a class.

To use this rule, a software engineer has to decide whether or not a given entity is relevant and autonomous. It is, however, not always clear whether this is the case or not. Aksit and Marcelloni have proposed to use fuzzy logic to enable the software engineer to provide uncertain input, like "the entity is fairly relevant and partly autonomous", instead of having to take premature yes-no decisions.

In this paper we introduce the concept of fuzzy booleans, and show that the use of fuzzy booleans in Zadeh's compositional rule of inference approach to approximate reasoning [9] is particularly suited to this kind of application.

Analogous to fuzzy numbers [3], which are fuzzy sets over the domain of numbers, we define fuzzy booleans to be fuzzy sets over the domain of truth-values {true,false}. A fuzzy boolean is denoted as (a,b), where a and b are numbers from the interval [0,1], a shorthand for the conventional notation "a/true + b/false". The truth-values 'true' and 'false' are represented by (1,0) and (0,1) respectively. The advantage of fuzzy booleans is that they allow fuzzy reasoning with concepts 'contradiction' and 'undefined'. For instance, when interpreted as possibilities, (1,1) is 'undefined', and (0,0) is 'contradiction'; when interpreted as necessities, this is just the other way around.

In order to demonstrate the usefulness of fuzzy booleans, we will prove two properties, which are valid in general, but which are of practical interest in the case of fuzzy booleans, as they involve sets of fuzzy rules whose size is equal to the size of the domain of discourse. The first property is that each set of fuzzy rules is equivalent to a set of rules whose antecedents are crisp. We will show how this set of rules can be derived from the original set of rules. The second property is that approximate reasoning
using a set of fuzzy rules with crisp antecedents and crisp consequents is equivalent with application of Zadeh's extension principle [8,10], irrespective whether the interpolation [5] or the implication method [4] is adopted, and irrespective of the particular t-norm or implication operator. As a consequence, in some cases the approximate reasoning process can be replaced by application of Zadeh's extension principle, leading to a significant increase in efficiency.

The usefulness of our results will be demonstrated by applying them to the aforementioned area of object-oriented design methods. We will show how approximate reasoning with fuzzy booleans can be applied to the problem of handling uncertain input, thereby improving the results of Aksit and Marcelloni [1,6,7].

2 Preliminaries

There are two variants of approximate reasoning with the compositional rule of inference: the interpolation method [5], and the implication method [4]. These two variants have been compared in [2].

Given universes X and Y, a fuzzy set A on X and a fuzzy relation R on X \times Y, the result of the compositional rule of inference is the fuzzy set B on Y defined by

\[ \forall y \in Y: B(y) = \sup_x \min (A(x), R(x,y)) \]  

(1)

The relation R is determined by a set of N fuzzy rules of the form

IF X = A_i THEN Y = B_i  

(2)

where A_i and B_i are fuzzy sets on X and Y respectively, for 1 \leq i \leq N.

Each rule determines a relation R_i, and the N relations R_i together determine the relation R.

In Mamdani's interpolation method, R_i is given by

\[ \forall x \in X, \forall y \in Y: R_i(x,y) = T(A_i(x), B_i(y)) \]  

(3)

where T is a t-norm. A t-norm is a function of type \([0,1] \times [0,1] \Rightarrow [0,1]\) which satisfies T(0,0) = T(0,1) = T(1,0) = 0, T(1,1) = 1, and some additional smoothness axioms (see e.g. [4]). In this paper, we assume the following properties of t-norms:

\[ \forall x \in [0,1]: T(0,x) = x \]  

(4a)

\[ \forall x \in [0,1]: T(1,x) = x \]  

(4b)

Usually T is chosen to be the minimum operator: T(x,y) = \min(x,y), but sometimes other t-norms are used.

The relation R is defined to be the union of all R_i:

\[ \forall x \in X, \forall y \in Y: R(x,y) = \max_i T(A_i(x), B_i(y)) \]  

(5)

In the implication method, R_i is given by

\[ \forall x \in X, \forall y \in Y: R_i(x,y) = J(A_i(x), B_i(y)) \]  

(6)

where J is an implication operator. An implication operator is a function of type \([0,1] \times [0,1] \Rightarrow [0,1]\) which satisfies J(0,0) = J(0,1) = J(1,1) = 1, J(1,0) = 0, and some additional smoothness axioms (see e.g. [4]). In this paper, we assume the following properties of implication operators:

\[ \forall x \in [0,1]: J(0,x) = 1 \]  

(7a)

\[ \forall x \in [0,1]: J(1,x) = x \]  

(7b)

Note that from the extensive list of implication operators in [4], all except one (Willmott implication) satisfy these properties.

The relation R is defined to be the intersection of all R_i:

\[ \forall x \in X, \forall y \in Y: R(x,y) = \min_i J(A_i(x), B_i(y)) \]  

(8)

3 Equivalence of sets of fuzzy rules

Two sets of fuzzy rules are said to be equivalent if they determine the same relation. Equivalence of sets of fuzzy rules depends on the chosen approach, and on the chosen t-norm or implication operator. For both approaches, for each t-norm or implication operator, and for each set of fuzzy rules, such as given by eq. (2), we will derive an equivalent set of fuzzy rules with crisp singleton antecedents.

First we consider the implication method. Let a set of fuzzy rules be given; the corresponding relation is given by eq. (8). Let a be an element of the universe X, and consider the fuzzy rule where the antecedent is the crisp singleton set containing a:

IF X = \{1/a\} THEN Y=B_a  

(9)

The relation R_a which is determined by this rule is given by:

\[ \forall y \in Y: R_a (a,y) = J (1, B_a(y)) = B_a(y) \]  

(10a)

\[ \forall x \in X, \forall y \in Y: x \neq a \Rightarrow R_a (x,y) = J(0, B_a(y)) = 1 \]  

(10b)
Now consider the set of all such fuzzy rules for each \( a \) in \( X \). The relation \( R \) determined by this set is the intersection of the relations \( R_a \):

\[
\forall x \in X, \forall y \in Y : R(x,y) = \min_a R_a(x,y) = B^i(y) \tag{11}
\]

Comparing with eq. (8), we find that this relation equals the relation of the set of fuzzy rules in eq. (2) when the consequents in eq. (9) are defined by

\[
B^i(y) = \min_i J(A_i(x),B_i(y)) \tag{12}
\]

Next consider the interpolation method. The relation is given by eq. (5). The analogon of eq. (10) is

\[
\forall y \in Y: R_a(a,y) = T(1,B^i(y)) = B^a(y) \tag{13a}
\]

\[
\forall x \in X, \forall y \in Y : x \neq a \Rightarrow R_a (x,y) = T(0,B^i(y)) = 0 \tag{13b}
\]

and the analogon of eq. (11) is

\[
\forall x \in X, \forall y \in Y : R(x,y) = \max_a R_a(x,y) = B^*(y) \tag{14}
\]

which is equal to eq. (5) when \( B^* \) is defined by:

\[
B^*(y) = \max_i T(A_i(x),B_i(y)) \tag{15}
\]

So, for both variants of approximate reasoning with the compositional rule of inference, each set of fuzzy rules is equivalent to a set of fuzzy rules with crisp singleton antecedents. In general, this fact will be of limited value, as the size of the equivalent set of fuzzy rules is equal to the size of the universe \( X \). However, in the case of fuzzy boolean antecedents, the size of the universe equals 2. This means that, when using fuzzy boolean antecedents, no more than 2 rules are needed, and the antecedents of these rules are just the crisp values true and false:

IF \( X = (1,0) \) THEN \( Y = B^{\text{true}} \) \hspace{1cm} (16a)

IF \( X = (0,1) \) THEN \( Y = B^{\text{false}} \) \hspace{1cm} (16b)

where \( B^{\text{true}} \) and \( B^{\text{false}} \) are fuzzy sets on the domain \( Y \).

In case the rules have \( N \) antecedents, the cartesian product of these antecedents is considered as a single antecedent on the universe which is the cartesian product of \( N \) copies of the universe \{true,false\}. Each set of rules then is equivalent to a set of \( 2^N \) rules with singleton crisp antecedents.

It is worthy to observe another remarkable fact here. Suppose that we have transformed our set of fuzzy rules to a set of rules with crisp singleton antecedents, as described above. The relation determined by this new set is given by either eq. (11) or eq. (14). This relation does not depend on the approach, nor on the choice of \( t \)-norm or implication operator. This means that, after the transformation, we need not bother about the choice of approach and the choice of \( t \)-norm or implication operator, since the derived relation is the same in all cases, and so the inference results are the same in all cases. This is true in case our set of rules is complete, i.e. there is a rule for each element in the universe \( X \). So possibly we have to add "dummy" rules, i.e. rules with empty consequent in case of the interpolation method, and rules whose consequent is the whole universe in case of the implication method. The same result holds if the antecedents of the rules are crisp sets, not necessarily singleton sets, which form a partition of the universe \( X \).

### 4 Approximate reasoning with fuzzy booleans

In this section we will consider approximate reasoning with fuzzy boolean antecedents in detail. We consider first the most simple case, where there is only one fuzzy boolean antecedent. Given a set of fuzzy rules, it can be transformed, as described in the previous section to a complete set of rules as given in eq. (16). The consequents, \( B^{\text{true}} \) and \( B^{\text{false}} \), depend on the original rules, the approach (interpolation or implication method), and the \( t \)-norm or implication operator. The corresponding relation, and thus all inferences, are independent of the approach, and independent of \( t \)-norm or implication operator. The relation is given by

\[
\forall y \in Y : R(\text{true},y) = B^{\text{true}}(y) \tag{17a}
\]

\[
\forall y \in Y : R(\text{false},y) = B^{\text{false}}(y) \tag{17b}
\]

Let the input be the fuzzy boolean \( (a,b) \).

Then the inference result \( B \), using the compositional rule of inference, is given by

\[
\forall y \in Y : B(y) = \sup_x \min \left( (a,b)(x), R(x,y) \right) \tag{18}
\]

where the supremum is over the truth-values, and \( (a,b)x \) means the membership value of \( x \) in the fuzzy set \( (a,b) \), so \( (a,b)(\text{true}) = a \) and \( (a,b)(\text{false}) = b \). Using eq. (17), it follows that

\[
\forall y \in Y : B(y) = \max \left( \min (a, B^{\text{true}}(y)), \min (b, B^{\text{false}}(y)) \right) \tag{19}
\]
Before considering the most general case, consider the case of two fuzzy boolean antecedents. Each set of fuzzy rules can be transformed into a complete set of four fuzzy rules:

- \( \text{IF } X_1 = (1,0) \text{ AND } X_2 = (1,0) \text{ THEN } Y = B_{\text{true,true}} \) \hspace{1cm} (20a)
- \( \text{IF } X_1 = (1,0) \text{ AND } X_2 = (0,1) \text{ THEN } Y = B_{\text{true,false}} \) \hspace{1cm} (20b)
- \( \text{IF } X_1 = (0,1) \text{ AND } X_2 = (1,0) \text{ THEN } Y = B_{\text{false,true}} \) \hspace{1cm} (20c)
- \( \text{IF } X_1 = (0,1) \text{ AND } X_2 = (0,1) \text{ THEN } Y = B_{\text{false,false}} \) \hspace{1cm} (20d)

If the inputs are \( X_1 = (a,b) \) and \( X_2 = (c,d) \), then the inference result is

\[
\forall y \in Y : B(y) = \max \left\{ \min (a,c, B_{\text{true,true}}(y)), \min (a,d, B_{\text{true,false}}(y)), \min (b,c, B_{\text{false,true}}(y)), \min (b,d, B_{\text{false,false}}(y)) \right\} \hspace{1cm} (21)
\]

Denoting the inputs by \( X_1 = A_1 \) and \( X_2 = A_2 \), this can be written as

\[
\forall y \in Y : B(y) = \max \left\{ \min (A_1(b_1), A_2(b_2), B_{b_1,b_2}(y)) \mid b_1,b_2 \in \{\text{true,false}\} \right\} \hspace{1cm} (22)
\]

Written in this way, generalisation to \( N \) fuzzy boolean antecedents is easy:

\[
\forall y \in Y : B(y) = \max \left\{ \min (A_1(b_1), \ldots, A_N(b_N), B_{b_1,\ldots,b_N}(y)) \mid b_1,\ldots,b_N \in \{\text{true,false}\} \right\} \hspace{1cm} (23)
\]

From a computational point of view, the time needed for an inference with the formula above is exponential in \( N \). This is not considered to be a problem, however, since \( N \), the number of antecedents, usually is a small number.

### 5 Zadeh's extension principle

In this section we will show that Zadeh's extension principle can be derived as a consequence of the formalism of approximate reasoning, using a complete set of rules for which both antecedents and consequents are crisp singleton sets. As in the previous section, this general result will take an elegant form in the case of fuzzy booleans.

Let \( f \) be a function from the universe \( X \) onto the universe \( Y \). By Zadeh's extension principle, \( f \) maps each fuzzy set \( A \) on \( X \) onto a fuzzy set \( B \) on \( Y \) which is given by

\[
B(y) = \sup_{x:f(x)=y} A(x) \hspace{1cm} (24)
\]

Consider, for some \( a \in X \), the following rule whose antecedent and consequent are both singleton sets:

\[
\text{IF } X = \{1/a\} \text{ THEN } Y = \{1/f(a)\} \hspace{1cm} (25)
\]

Assuming the interpolation method, this rule determines the relation

\[
R_a(a,f(a)) = 1 \hspace{1cm} (26a)
\]
\[
\forall x \in X, \forall y \in Y : (x,y) \neq (a,f(a)) \Rightarrow R_a(x,y) = 0 \hspace{1cm} (26b)
\]

Now consider the collection of rules of this form for all \( a \in X \).

This complete set of rules determines the relation \( R \) which is given by

\[
\forall x \in X, \forall y \in Y : R(x,y) = \sup_a R_a(x,y) \hspace{1cm} (27)
\]

From eqs. (26) and (27) it follows that

\[
\forall x \in X, \forall y \in Y : (x,y) \neq f(x) \Rightarrow R(x,y) = 0 \hspace{1cm} (28a)
\]
\[
\forall x \in X, \forall y \in Y : y \neq f(x) \Rightarrow R(x,y) = 0 \hspace{1cm} (28b)
\]

Given fuzzy set \( A \) on \( X \), approximate reasoning with this collection of rules gives the fuzzy set \( B \) on \( Y \) given by

\[
\forall y \in Y : B(y) = \sup_x \min (A(x),R(x,y)) = \sup_{x:f(x)=y} A(x) \hspace{1cm} (29)
\]

which is the same result as the result from Zadeh's extension principle (eq. (24)).

From the result of the previous section it follows that the result for the implication method is the same.

As in the previous section, this result is of limited value in the general case, as the number of rules is equal to the size of the universe \( X \), which generally is very large or infinite. In the case of fuzzy booleans, however, the universe \( X \) generally is quite small. For instance, to express that the fuzzy boolean \( Y \) is the negation of the fuzzy boolean \( X \), two rules suffice:

\[
\text{IF } X = (1,0) \text{ THEN } Y = (0,1) \hspace{1cm} (30a)
\]
\[
\text{IF } X = (0,1) \text{ THEN } Y = (1,0) \hspace{1cm} (30b)
\]

Likewise, the following 4 rules express that \( Z \) equals \( X \) AND \( Y \):

\[
\text{IF } X = (1,0) \text{ AND } Y = (1,0) \text{ THEN } Z = (1,0) \hspace{1cm} (31a)
\]
\[
\text{IF } X = (1,0) \text{ AND } Y = (0,1) \text{ THEN } Z = (0,1) \hspace{1cm} (31b)
\]
\[
\text{IF } X = (0,1) \text{ AND } Y = (1,0) \text{ THEN } Z = (0,1) \hspace{1cm} (31c)
\]

\[
\text{IF } X = (0,1) \text{ AND } Y = (0,1) \text{ THEN } Z = (0,1) \hspace{1cm} (31d)
\]
6 Application to object-oriented design methods

In this section we will show how approximated reasoning with fuzzy booleans can be applied in the field of object-oriented design methods. In a series of papers [1,6,7], Aksit and Marcelloni have considered the problem of object-oriented design methods in the presence of uncertainty. They consider design rules of the form:

IF an entity is relevant AND it is autonomous
THEN it is a class.

The problem here is that it is in general not certain whether a given entity is relevant, or whether it is autonomous. Their approach is to introduce linguistic values: weakly, slightly, fairly, substantially and strongly for relevancy and classhood, and dependent, partially dependent and fully autonomous for autonomy. These linguistic values are associated with fuzzy numbers, i.e. fuzzy sets whose domain is the interval [0,1]. The rule given above is replaced by a set of 15 rules, each of the form

IF an entity is X relevant AND it is Y autonomous THEN it is Z a class.

Here X, Y and Z are linguistic values, and the 15 rules are given in the following table, which lists Z, given X (horizontally) and Y (vertically)

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Using these rules, the classhood of an entity can be determined as a fuzzy set on the interval [0,1], given the fuzzy sets which describe its relevancy and its autonomy.

This approach has several drawbacks.

1. The original rule and the set of fuzzy rules do not have the same meaning. Where the original rule is an implication, meaning that when the entity is not relevant and/or not autonomous, there is no information about its classhood. The set of fuzzy rules, on the contrary, indicate that in that case the entity is not ("weakly") a class.

2. Using fuzzy sets on the interval [0,1] as values for fuzzy relevancy of an entity suggests that the crisp values for relevancy are real numbers from [0,1]. However, since an entity is either relevant or not relevant, the crisp values for relevancy are the truth values true and false. The same holds for autonomy and classhood.

3. It seems unreasonable to have to devise a set of fuzzy rules from the given rule, as all information is present in the rule, and no extra information is available.

4. The Aksit-Marcelloni approach does not consider the semantics of the fuzzy rules. In particular, a choice between the interpolation method and the implication method is not made; also a choice of t-norm or implication operator is not made.

The obvious alternative, which solves all these drawbacks, is to use fuzzy booleans. It seems natural that the fuzzy values associated to the concepts relevancy, autonomy and classhood are fuzzy booleans. Moreover, no set of fuzzy rules need to be devised; it suffices to use the given rule, which can be reformulated as:

IF relevance = (1,0) AND autonomy = (1,0)
THEN classhood = (1,0)

Since there is no complete set of fuzzy rules, the results for the interpolation method and the implication method will not be the same. Using the interpolation method, with any t-norm, inference with the rule above is as follows. Given relevance = (a,b), autonomy = (c,d), it is inferred that classhood = (min(a,c),0). Note that the membership value of false in classhood is always zero. If either a=0 or c=0, the result for classhood is (0,0), meaning that classhood is undefined. Using the implication method, with any implication operator, the inferred classhood is (max (min(a,c),min(a,d),min(b,c), min(b,d)), max (min(a,d),min(b,c) min(b,d))).

If (a,b)=(1,0) and (c,d)=(1,0) this equals (1,0), but if one of the inputs (or both) is changed into (0,1), this equals (1,1), meaning that classhood is undefined.

This result is different from the result of the Aksit-Marcelloni approach, which gives a classhood which is a fuzzy set on [0,1], with non-zero membership values in the vicinity of 0 (the exact form depends on the choice of fuzzy sets for weakly, slightly, etc.). Defuzzification of this fuzzy set would give a value in the neighbourhood of 0, indicating that classhood is (approximately) false. This result of the Aksit-
Marcelloni approach can be obtained via our approach by adding 3 more rules:
IF relevance = (1,0) AND autonomy = (0,1)
    THEN classhood = (0,1)
IF relevance = (0,1) AND autonomy = (1,0)
    THEN classhood = (0,1)
IF relevance = (0,1) AND autonomy = (0,1)
    THEN classhood = (0,1)

Since now the set of fuzzy rules is complete, the inference results for the interpolation method and the implication method are the same. Given relevance = (a,b), autonomy = (c,d), the inferred fuzzy set for classhood is (min(a,c), max(min(a,b),min(b,c),min(b,d))).

In accordance with the results of section 3, the same fuzzy set for classhood is obtained from
classhood = relevance AND autonomy
where AND is obtained from the Boolean and-operator via Zadeh's extension principle.

By considering a universe with 2 elements instead of a continuum, our approach is computationally far more efficient than the Aksit-Marcelloni approach, even if in the latter membership functions are restricted to be piecewise linear. Resorting to Zadeh's extension principle even increases the efficiency. On the other hand, our approach has all advantages of the Aksit-Marcelloni approach, such as deferring of design decisions and modelling of inconsistencies.

References


